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LETTER TO THE EDITOR

Cluster shapes at the percolation threshold: an effective cluster dimensionality and its connection with critical-point exponents†

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Abstract. An effective dimensionality d_p is introduced for the purpose of providing a quantitative characterisation of the degree of ramification of the clusters that occur at the percolation threshold. It is found that d_p is directly related to percolation critical exponents, and that $1 \leqslant d_p \leqslant d$, which in turn places bounds on certain scaling powers and critical exponents. The exponents—when renormalised according to Suzuki's 'extended universality' prescription—have an appealingly simple form in terms of d_p; in particular, the renormalised mean cluster size exponent is d_p, while both the order parameter and 'decay of correlation' exponents are given by the co-dimension $d - d_p$.

1. Introduction

The possibility that cluster shapes near the critical point are characterised by a high degree of ramification or 'stringiness' has been discussed frequently in recent work (see, e.g., Domb *et al* 1975, Stauffer 1975, Quinn *et al* 1976, Binder 1976, Temperley 1976, Leath 1976, Domb 1976). However the precise significance of this concept is yet to be clarified, and attempts to place the qualitative feature of ramification on a fully quantitative footing have been fraught with problems. It is plausible that some features of cluster shape may be directly related to aspects of observed critical phenomena; therefore it is desirable to identify certain properties of cluster shape that have precise geometric meaning and are related unambiguously to the quantitative parameters currently used to characterise the critical point. In this Letter we propose such a property, an 'effective cluster dimensionality' d_p‡.

We focus our discussion here on cluster shapes near the connectivity threshold, the 'critical point' of the percolation problem; the generalisation to other systems is under investigation. To a first approximation, the percolation problem is a useful description of the condensation of f-functional monomers at the gelation threshold (Flory 1953 and references therein). In particular, the sudden condensation of a single gel molecule—distinguished from the molecules of the sol phase by being essentially infinite in spatial extent—is analogous to the appearance of a single infinite cluster at p_c in the

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‡ So far as we can tell, d_p is equivalent to the Hausdorff–Besicovitch dimensionality (Hausdorff 1919, Besicovitch 1935), a mathematical concept further developed and extensively applied to a wide range of systems by Mandelbrot (1975); Mandelbrot calls objects with non-integer Hausdorff–Besicovitch dimensionality 'fractals', and the Hausdorff–Besicovitch dimensionality itself he terms the 'fractal dimensionality'.

percolation problem. Thus to the extent that this analogy is valid, we may interpret d_p as an 'effective polymer dimensionality' at the gelation threshold.

We will find that d_p is significantly smaller than d, the system dimensionality, which is consistent with the qualitative notion that clusters near the critical point are highly ramified. In particular, we find $d_p = 2$ for the Flory theory of gelation, implying that in this closed form theory the molecules are effectively two dimensional even though they may 'fill' a three-dimensional space.

2. Definition of d_p

If the incipient infinite cluster at p_c had the full connectivity of a d-dimensional lattice, then we would expect that for large finite clusters very near p_c, $\chi_p \sim \xi_p^d$; here χ_p denotes the mean number of occupied sites belonging to a cluster, and ξ_p denotes the mean linear dimension of a cluster†. For highly ramified clusters, on the other hand, we expect $\chi_p \ll \xi_p^d$ (cf figure 1). It is of interest to enquire whether there exists a number d_p such that

$$\chi_p \sim (\xi_p)^{d_p}. \tag{1a}$$

Indeed, such a number would seem to exist, since from the definitions of the critical exponents γ_p and ν_p characterising the divergence at p_c of χ_p and ξ_p respectively, we have

$$d_p = \gamma_p / \nu_p. \tag{1b}$$

Note that the 'effective cluster dimensionality' d_p defined in (1a) should depend only on d since percolation exponents depend only on d.

3. Dependence of d_p on d

For $d = 1$ and $d = \infty$, we know the critical exponents exactly and hence we can calculate d_p exactly‡. We find

$$d_p(1) = 1, \tag{2a}$$

which is not surprising, and

$$d_p(\infty) = 2, \tag{2b}$$

† We use the definitions of percolation functions and corresponding critical exponents exactly as in Essam (1972) with one exception: we use the notation χ_p instead of $S(p)$ in order to stress the parallel with thermal phase transitions—for example, for $d = 1$, $\chi_p = (1+p)/(1-p)$ while $\chi_T = (1+y)/(1-y)$, where χ_T is the isothermal susceptibility and $y \equiv \tanh(J/kT)$ (Reynolds *et al* 1977, Klein *et al* 1977, Stauffer and Jayaprakash 1977, Stanley *et al* 1977).

‡ For $d = 1$, $p_c = 1$, $\chi_p \sim (1+p)/(1-p)$, and $\xi_p^2 = 2p/(1-p)^2$; hence $\gamma_p = \nu_p = 1$. The lattice dimensionality of a Cayley tree is not well defined (it cannot be embedded in any finite-dimensional space), but it is believed that percolation exponents for the interior of a Cayley tree (Flory 1953, Fisher and Essam 1961) are those of an infinite-dimensional system. For a Cayley tree, there are no closed cycles, and the Flory theory of the gelation threshold becomes exact for this lattice; consequently the dimensionality of the incipient gel molecule is two, regardless of the system dimensionality (cf figure 2).

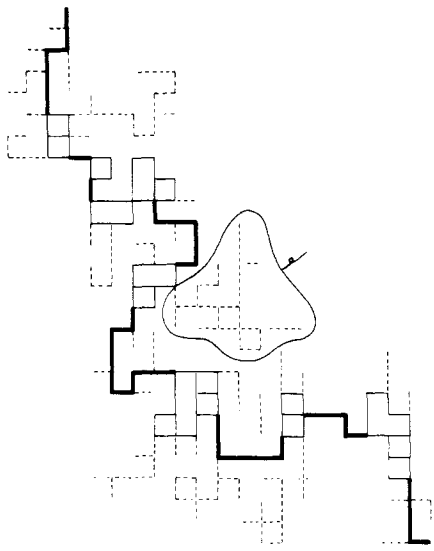

Figure 1. Computer-generated 'incipient infinite cluster' for the bond percolation problem on a 25×20 square lattice with $p=p_c=0{\cdot}5$. Note the qualitative feature of 'ramification' or 'stringiness' of this cluster, corresponding to the fact that the effective cluster dimensionality d_p is less than d (for $d=2$, $d_p\simeq1{\cdot}8$). If we think of this cluster as representing a dilute magnetic system in which only a fraction p of the exchange integrals are non-zero, then it is clear that magnetic correlations will spread from one end of the lattice to another along a path that is dominated by the 'backbone bonds' (shown as full lines) rather than by the 'dangling ends' (shown as broken lines). The backbone bonds that are singly-connected are shown as bold lines while those that are multiply connected are shown as light lines. By construction, the bold lines form self-avoiding walks, and it has been conjectured that 'globally' the backbone is a SAW interpreted by 'blobs' of multiply-connected bonds. In support of this conjecture, d_p is identical for both the percolation problem and the SAW problem.

which is surprising since it seems to imply that in the limit of infinite lattice dimensionality the effective cluster dimensionality is only two!

What do we expect for $1<d<\infty$? For $d=1$, our intuitive understanding of equation (2*a*) is that the connectivity necessary for percolation is achieved only for $p=1$, and the percolation cluster therefore has $d_p=d$. For $d>1$, we expect that the percolation cluster may have $d_p<d$, since percolation is achieved at $p_c<1$. We also expect $d_p>1$, since the cluster shape at p_c will be considerably more 'branched' than a simple linear chain. Thus for $d>1$, we expect that $1<d_p<d$.

Although no exact results for percolation exponents are known for $1<d<\infty$, considerable information on cluster geometry has been obtained using the methods of low-density and high-density series expansions, direct computer simulations, position-space renormalisation group, and renormalisation group expansions in the parameter $(6-d)$. The relevant results of this work are summarised in table 1. From the limited information on γ_p and ν_p it is impossible to obtain accurate estimates of d_p, especially for $d=4$, 5. However, if we make the plausible assumption that weak scaling holds (Essam and Gwilym 1971), then we can use the additional expression

$$d_p=(2-\alpha_p-2\beta_p)/\nu_p. \tag{3}$$

Table 1. Summary of existing calculations that can be used to obtain numerical values for $d_p(d)$. Each line of the table represents a different estimate, obtained using the method shown in the first column and the critical exponent values shown in subsequent columns. Here PSRG denotes 'position space renormalisation group'.

Method and equation used	γ_p	ν_p	α_p	β_p	δ_p	d_p
$d=2$						
series, equation (1*b*)	$2{\cdot}38\pm0{\cdot}02$[a]	$1{\cdot}34\pm0{\cdot}02$[a]	—	—	—	$1{\cdot}78\pm0{\cdot}04$
(1*b*)	$2{\cdot}43\pm0{\cdot}03$[b]	$1{\cdot}34\pm0{\cdot}02$[a]	—	—	—	$1{\cdot}81\pm0{\cdot}05$
(1*b*)	$2{\cdot}43\pm0{\cdot}03$[b]	$1{\cdot}32^{+0{\cdot}02}_{-0{\cdot}07}$[c]	—	—	—	$1{\cdot}84^{+0{\cdot}13}_{-0{\cdot}05}$
(1*b*)	$2{\cdot}38\pm0{\cdot}02$[a]	$1{\cdot}32^{+0{\cdot}02}_{-0{\cdot}07}$[c]	—	—	—	$1{\cdot}80^{+0{\cdot}12}_{-0{\cdot}04}$
(3)	—	$1{\cdot}34\pm0{\cdot}02$[a]	$-0{\cdot}668\pm0{\cdot}004$[d]	$0{\cdot}138\pm0{\cdot}007$[e]	—	$1{\cdot}785\pm0{\cdot}04$
(3)	—	$1{\cdot}32^{+0{\cdot}02}_{-0{\cdot}07}$[c]	$-0{\cdot}668\pm0{\cdot}004$[d]	$0{\cdot}138\pm0{\cdot}007$[e]	—	$1{\cdot}81^{+0{\cdot}12}_{-0{\cdot}04}$
(4*a*)	$2{\cdot}38\pm0{\cdot}02$[a]	—	$-0{\cdot}668\pm0{\cdot}004$[d]	—	—	$1{\cdot}784\pm0{\cdot}02$
(4*a*)	$2{\cdot}43\pm0{\cdot}03$[b]	—	$-0{\cdot}668\pm0{\cdot}004$[d]	—	—	$1{\cdot}82\pm0{\cdot}02$
(4*b*)	$2{\cdot}38\pm0{\cdot}02$[a]	—	—	$0{\cdot}138\pm0{\cdot}007$[e]	—	$1{\cdot}792\pm0{\cdot}04$
(4*b*)	$2{\cdot}43\pm0{\cdot}03$[b]	—	—	$0{\cdot}138\pm0{\cdot}007$[e]	—	$1{\cdot}796\pm0{\cdot}05$
(4*c*)	—	—	$-0{\cdot}668\pm0{\cdot}004$[d]	$0{\cdot}138\pm0{\cdot}007$[e]	—	$1{\cdot}793\pm0{\cdot}016$
(4*d*)	—	—	—	—	$18{\cdot}0\pm0{\cdot}75$[f]	$1{\cdot}79^{+0{\cdot}16}_{-0{\cdot}14}$
computer simulations	—	—	—	—	—	$1{\cdot}78$[g]
PSRG: square; bond	—	—	—	—	$18{\cdot}249$[h]	1·792
PSRG: square; bond	—	—	—	—	—	$1{\cdot}9$[i]
PSRG: triangular; site	—	—	—	—	—	$1{\cdot}68$[i]
RG (Migdal–Kadanoff): square; bond	—	—	—	—	—	$1{\cdot}78$[j]
ϵ expansion (5)	—	—	—	—	—	2·55
$d=3$						
series, equation (1*b*)	$1{\cdot}66\pm0{\cdot}07$[k]	$0{\cdot}83+1{\cdot}5\Delta p_c\pm0{\cdot}01$[a]	—	—	—	$2{\cdot}00\pm0{\cdot}11$
(1*b*)	$1{\cdot}66\pm0{\cdot}07$[k]	$0{\cdot}825\pm5{\cdot}0\Delta p_c\pm0{\cdot}02$[c]	—	—	—	$2{\cdot}01\pm0{\cdot}22$
(1*b*)	$1{\cdot}70\pm0{\cdot}11$[a]	$0{\cdot}825+5{\cdot}0\Delta p_c\pm0{\cdot}02$[c]	—	—	—	$2{\cdot}06\pm0{\cdot}27$
(4*b*)	$1{\cdot}66\pm0{\cdot}07$[k]	—	—	$0{\cdot}42\pm0{\cdot}06$[l]	—	$1{\cdot}992^{+0{\cdot}25}_{-0{\cdot}23}$
(4*b*)	$1{\cdot}70\pm0{\cdot}11$[a]	—	—	$0{\cdot}42\pm0{\cdot}06$[l]	—	$2{\cdot}008^{+0{\cdot}34}_{-0{\cdot}30}$
(4*d*)	—	—	—	—	$5{\cdot}0\pm0{\cdot}8$[m]	$2{\cdot}00^{+0{\cdot}76}_{-0{\cdot}59}$
computer (4*b*)	$1{\cdot}8\pm0{\cdot}005$[n]	—	—	$0{\cdot}39\pm0{\cdot}02$[n]	—	$2{\cdot}09^{+0{\cdot}14}_{-0{\cdot}12}$
ϵ expansion (5)	—	—	—	—	—	2·34
$d=4$						
series (4*b*)	$1{\cdot}41\pm0{\cdot}25$[o]	—	—	$0{\cdot}52\pm0{\cdot}03$[n]	—	$2{\cdot}30^{+0{\cdot}80}_{-0{\cdot}62}$
computer (4*b*)	$1{\cdot}6\pm0{\cdot}1$[n]	—	—	$0{\cdot}52\pm0{\cdot}03$[n]	—	$2{\cdot}43^{+0{\cdot}32}_{-0{\cdot}28}$
ϵ expansion (5)	—	—	—	—	—	2·18

Table 1.—*continued*

Method and equation used		γ_p	ν_p	α_p	β_p	δ_p	d_p
$d=5$							
series	(4*b*)	$1{\cdot}25\pm0{\cdot}15$[o]	—	—	$0{\cdot}66\pm0{\cdot}05$[n]	—	$2{\cdot}43\{^{+0.58}_{-0.48}$
computer	(4*b*)	$1{\cdot}3\pm0{\cdot}1$[n]	—	—	$0{\cdot}66\pm0{\cdot}05$[n]	—	$2{\cdot}48\{^{+0.41}_{-0.36}$
ϵ expansion	(5)	—	—	—	—	—	2·07
$d=6$							
series	(4*b*)	$1{\cdot}06\pm0{\cdot}20$[o]	—	—	$0{\cdot}97\pm0{\cdot}05$[n]	—	$2{\cdot}12\{^{+0.68}_{-0.56}$
computer	(4*b*)	$1{\cdot}00\pm0{\cdot}05$[n]	—	—	$0{\cdot}97\pm0{\cdot}05$[n]	—	$2{\cdot}04\{^{+0.22}_{-0.20}$
ϵ expansion	(5)	—	—	—	—	—	2·0

[a] Dunn *et al* (1975): *b*
[b] Sykes *et al* (1976b): *b,s*
[c] Cox and Essam (1976): *s*
[d] Domb and Pearce (1976): *b*
[e] Sykes *et al* (1976c): *b,s*
[f] Gaunt and Sykes (1976): *s,b*
[g] Mandelbrot (1977): *s*
[h] Dasgupta (1976): *b*
[i] Reynolds *et al* (1977): *b,s*
[j] Riedel *et al* (1977): *b*
[k] Sykes *et al* (1976d): *s*
[l] Sykes *et al* (1976*a*): *b,s*
[m] Gaunt (1977): *s*
[n] Kirkpatrick (1976): *s*
[o] Gaunt *et al* (1976): *s,b*

Further, if we believe that $d\nu_p \geqslant 2-\alpha_p$, then we obtain four additional expressions for d_p:

$$d_p \leqslant \begin{cases} d\gamma_p/(2-\alpha_p) & (4a) \\ d\gamma_p/(2\beta_p+\gamma_p) & (4b) \\ d(2-\alpha_p-2\beta_p)/(2-\alpha_p) & (4c) \\ d(\delta_p-1)/(\delta_p+1) & (4d) \end{cases}$$

where (4*d*) follows from the three-exponent equality $\gamma_p=\beta_p(\delta_p-1)$. All inequalities become equalities if strong scaling ('hyperscaling') holds, as is widely believed to occur at least for $d=2, 6$. The numerical data of table 1 are consistent with the result that $d_p<d$; in particular, it appears that $d_p(2)\simeq 1{\cdot}8$ and $d_p(3)\simeq 2$.

The situation for $d\simeq 6$ is particularly interesting. Firstly, Toulouse (1974) has conjectured that the Bethe lattice results hold for $6\leqslant d<\infty$, and his conjecture has received support from numerical work of Kirkpatrick (1976) and Gaunt *et al* (1976). Thus it is plausible that $d_p(d)=2$ for $d>6$. Harris *et al* (1975) have proposed that for $d\lesssim 6$,

$$d_p(\epsilon)=2+\epsilon/21+(206/3^3 7^3)\epsilon^2+O(\epsilon^3) \qquad (5)$$

where $\epsilon\equiv 6-d$; the order ϵ^2 term was calculated by Priest and Lubensky (1976) and Amit (1976). Since as d increases, the branching possibilities should not decrease, one would intuitively expect that d_p is a non-decreasing function of d‡. If this is the case, then $d_p(\infty)=2$ is the maximum value of d_p. Therefore equation (5) is surprising: although it predicts $d_p<d$ for all reasonable ϵ, it also predicts that the cluster dimensionality *increases* as the system dimensionality decreases (cf figure 2).

In summary, then, all available data are consistent with the conjecture that $d_p<d$.

‡ This intuition may be false, since $p_c(d)$ decreases with increasing d (P J Reynolds, private communication).

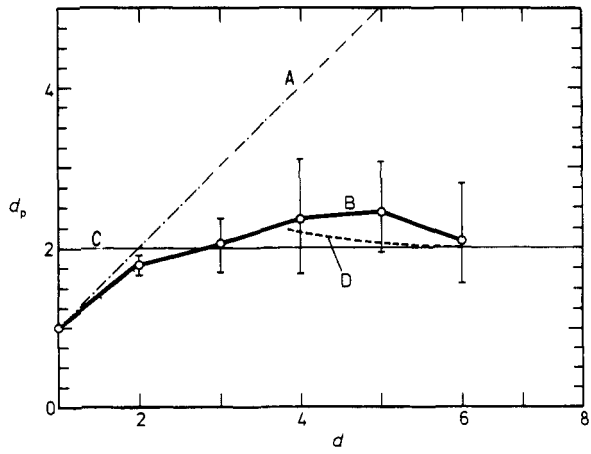

Figure 2. Shown are four possible dependences of d_p on d. A, if the incipient infinite cluster at p_c (or, equivalently, the incipient gel molecule) had the full connectivity of a d-dimensional lattice, then we would expect that $d_p = d$. B, for $d > 1$, the open circles represent averages of the numerical estimates of d_p from table 1 (and the 'error bars' indicate the range of the estimates), while for $d = 1$ there exists an exact solution. Note that $d_p \leqslant d$, and the difference $d - d_p$ (the co-dimension) is the Suzuki-renormalised order parameter exponent $2\beta_p/\nu_p$ and the 'decay of correlation' exponent $d - 2 + \eta_p$. C, the Flory theory of the gelation threshold predicts that $d_p = 2$ *for all* d; this result is correct in the limit $d \to \infty$, and probably correct for $d \geqslant 6$ (Toulouse 1974), but it is geometrically impossible for $d < 2$. Note that for $d = 3$, the Flory theory is in good agreement with numerical predictions. D, the $(6 - d)$ expansion proposed by Harris *et al* (1975) for $d \lesssim 6$.

4. Connection with the scaling hypothesis

The scaling hypothesis for the percolation functions which are analogous to thermodynamic functions was first formulated by Essam and Gwilym (1971). Here it is convenient to consider the scaling hypothesis directly for the pair connectedness $C_2(r, \Delta p, h)$, where h is the analogue of the magnetic field for the percolation problem, and $\Delta p \equiv p - p_c$. One assumes there exist three numbers b_q, b_p, and b_h such that for all positive λ,

$$S(\lambda^{b_q} q, \lambda^{b_p} \Delta p, \lambda^{b_h} h) = \lambda^{1 - d b_q} S(q, \Delta p, h) \tag{6}$$

where $S(q)$ is the spatial Fourier transform of $C_2(r)$. From (6) it follows that all critical exponents for functions derived from the pair connectedness can be expressed in terms of the three scaling powers b_q, b_p and b_h. In particular, one finds that $\nu_p = b_q/b_p$, $-\gamma_p = (1 - d b_q)/b_p$, and $2 - \eta_p = d - b_q^{-1}$ in complete analogy to the thermal problem (Hankey and Stanley 1972). Hence

$$d_p = 2 - \eta_p \tag{7}$$

and the expression $2 - \eta_p$ that occurs so frequently in relations among critical exponents is given a concrete interpretation in terms of a specific 'geometric' property of the percolation threshold. While $2 - \eta_p$ is the effective dimensionality, the length scaling power b_q^{-1} is the effective *co-dimensionality*,

$$b_q^{-1} = d - d_p. \tag{8}$$

The above relations follow directly from the 'weak scaling' assumption (6). If one further makes the 'strong scaling' assumption, then it follows that there are not three but only two independent scaling powers, since $b_h = db_q - \frac{1}{2}$ (Hankey and Stanley 1972). Hence we can also relate b_h to d_p: $2b_h = (d+d_p)/(d-d_p)$. Equivalently, $2y_h = d + d_p$, where $y_h = da_h$ is the usual magnetic field scaling power that occurs in thermodynamic scaling (a_h is defined precisely in Hankey and Stanley 1972). Therefore the inequality $1 \leqslant d_p \leqslant d$ leads to bounds on the scaling powers:

$$0 \leqslant b_q^{-1} \leqslant d-1, \qquad \frac{d+1}{d-1} \leqslant 2b_h \leqslant \infty, \qquad d+1 \leqslant 2y_h \leqslant 2d, \tag{9a}$$

which in turn lead to inequalities on the numerical values of the critical exponents that depend only on the length (or 'field') scaling powers:

$$2-d \leqslant \eta_p \leqslant 1 \qquad \text{and} \qquad \frac{d+1}{d-1} \leqslant \delta_p \leqslant \infty. \tag{9b}$$

Note that for the case $d = 1$, $1 \leqslant d_p \leqslant d$ becomes a strict equality, and we have, from (9), $b_q^{-1} = 0$, $b_h = \infty$, $y_h = 1$, $\eta_p = 1$, and $\delta_p = \infty$.

5. Connection with the extended universality hypothesis

Suzuki (1974) has pointed out that the apparent exceptions to the universality hypothesis are in fact not exceptions if one defines all critical exponents not through the variables q, H, and $\Delta T \equiv T - T_c$ but rather through the variables q, H, and $\kappa \sim (\Delta T)^{\nu}$. If we carry this procedure over to the percolation problem, equation (6) becomes

$$S(\lambda q, \lambda\kappa, \lambda^{b_h/b_q} h) = \lambda^{-d_p} S(q, \kappa, h) \tag{10}$$

and the number of independent scaling powers is reduced from three to two†. It is interesting to note that if we define a second 'fractal dimensionality' $d_\sigma = (2-\alpha_p)/\nu_p$, then both scaling powers and hence all critical exponents are readily expressible as simple functions of the two fractal dimensionalities d_σ and d_p. For example, the 'Suzuki-renormalised' order parameter (or 'weight fraction') exponent $\hat{\beta} \equiv 2\beta_p/\nu_p$ becomes $d_\sigma - d_p$, the mean size exponent $\hat{\gamma} = \gamma_p/\nu_p$ becomes d_p, the free energy exponent $\phi \equiv (2-\alpha_p)/\nu_p$ becomes d_σ, and the 'critical isotherm' exponents $\hat{\delta} \equiv \delta_p$ and $\hat{\eta} \equiv (d-2+\eta_p)$ become $(d_\sigma + d_p)/(d_\sigma - d_p)$ and $d - d_p$ respectively.

The numerical value of d_σ remains unspecified under the weak scaling assumption (one might conjecture $d_\sigma \leqslant d$ by analogy with the Josephson inequality, and $d_\sigma \geqslant d_p$ if $\beta_p \geqslant 0$). However, if we make the additional assumption necessary for 'strong scaling', then $b_h = db_q - \frac{1}{2}$; hence $d_\sigma = d$ and there is only a single independent parameter, d_p, in terms of which all renormalised critical exponents find remarkably simple expression: $\hat{\gamma} = d_p$, $\hat{\beta} = \hat{\eta} = d - d_p$ (the co-dimension), $\hat{\delta} = (d+d_p)/(d-d_p)$, and $\phi = d$.

Thus any two systems with identical values of d_p will have identical values of all 'renormalised' critical exponents provided strong scaling holds. For example, Suzuki observed that the eight-vertex model—which describes two independent Ising models on square lattices coupled by a four-spin interaction—has the same values of all renormalised critical exponents, independent of the coupling strength. This observation can be interpreted as the intuitively plausible statement that for this system the

† (10) follows on substituting $\kappa = (\Delta p)^{b_q/b_p}$ into (6), multiplying all scaling powers by b_q^{-1}, and using (8).

effective cluster dimensionality (or degree of cluster 'ramification') is independent of the coupling strength. Suzuki has also pointed out that the renormalised critical exponents are the same for the Ising model with triplet interactions as for the Ising model with pair interactions; this result can be interpreted to mean that both systems have the same effective cluster dimensionality. Finally, it should be pointed out that if two systems have the same value of d_p, there may be other common features. For example, Stanley *et al* (1976) have pointed out that d_p appears to be the same for the percolation and self-avoiding walk (SAW) problems, and this result has been interpreted to mean that correlations spread throughout a dilute magnet at the percolation threshold along paths that are, globally, SAW's (cf figure 1).

6. Discussion

Thus far our attention has focused on the particular definition of d_p given in equation (1). It is certainly legitimate to ask what would happen if we were to adopt differing definitions of d_p. Logically, we can alter equation (1) by changing either the length or the size functions therein. There is no *a priori* basis for choosing a 'best' definition of the effective cluster dimensionality, and definitions different than (1) may prove more suitable for certain applications. One alternative to (1)—which is equivalent to (1) when hyperscaling holds but gives larger numerical values for the effective cluster dimensionality—is obtained by focusing on the very large clusters. The typical length of the very large clusters diverges with the same power of (p_c-p), but the typical size s_L of the very large clusters may diverge faster than the mean size $\chi_p \equiv \langle s \rangle$ of all the finite clusters, thus leading to a different value of d_p. Information about s_L can be obtained by the following argument (Stauffer 1976): assume that the mean number of s-site clusters varies with s and p according to

$$n_s(p) \sim s^{-\tau} f((p_c-p)s^{1/\beta\delta}) \tag{11}$$

for very large s and very small $(p_c-p)/p_c$; then in some sense a typical size s_L of the very large clusters described by (11) may be expected to diverge as $s_L \sim (p_c-p)^{-\beta\delta}$. Note that for $d>1$, s_L grows much more rapidly as $p \to p_c$ than $\langle s \rangle$, since $\beta\delta = \beta+\gamma > \gamma$. Defining $d_p^\dagger$ through $s_L \sim \xi_p^{d_p^\dagger}$, we have†

$$d_p^\dagger = \beta_p\delta_p/\nu_p = d_p[\delta_p/(\delta_p-1)] \leqslant (d+d_p)/2. \tag{12}$$

If strong scaling holds, the inequality becomes an equality and $d_p^\dagger$ is fully equivalent to d_p, while if strong scaling does not hold, $d_p^\dagger$ and d_p are related by a critical point exponent. That the mean of a sub-group differs from the mean of the entire group is not surprising; what is perhaps intriguing is that as $p \to p_c$ the very large clusters increase in size sufficiently faster than the mean clusters that their effective dimensionality $d_p^\dagger$ is larger (for $d>1$) than the effective dimensionality d_p of the system as a whole.

It is straightforward to derive numerical values for $d_p^\dagger$ from the information given above and in table 1. In particular, we have $d_p^\dagger(d) \geqslant d_p(d)$, with the equality holding

† Very recently Harrison, Bishop and Quinn (private communication, and to be published) have obtained expression (12) for $d_p^\dagger$ using an elegant argument that does not explicitly involve the cluster-size scaling hypothesis. Moreover, they have confirmed (12) numerically for $d=2, 3$ by direct computer simulation of large clusters near p_c. Earlier $d=2$ data of Leath (1976) on smaller samples are also consistent with (12) (Leath interprets his data as being consistent with a fractal dimensionality of two).

for $d = 1$, and (from (5))

$$d_p^{\dagger}(\epsilon) = 4 - \frac{10}{21}\epsilon + \frac{103}{3^3 7^3}\epsilon^2 + \ldots \tag{13}$$

which results in the *monotonic* sequence $d_p^{\dagger}(\epsilon) = 4, 3{\cdot}53, 3{\cdot}09, 2{\cdot}67$ for $d = 6, 5, 4, 3$ respectively. If strong scaling holds, then $2b_h = d_p^{\dagger}/(d - d_p^{\dagger})$ and $y_h = d_p^{\dagger}$, while if strong scaling does not hold, no scaling power is given in terms of $d_p^{\dagger}$ alone. The 'Suzuki-renormalised' critical exponents, when expressed in terms of $d_p^{\dagger}$, are $\hat{\beta} = \hat{\eta} = 2(d - d_p^{\dagger})$, $\hat{\gamma} = 2d_p^{\dagger} - d$, and $\hat{\delta} = d_p^{\dagger}/(d - d_p^{\dagger})$.

In conclusion, then, we have proposed an effective cluster dimensionality d_p that describes the shape of the clusters (molecules) present at the percolation (gelation) threshold. We showed that d_p is directly related to critical exponents and we then used existing information about these exponents to evaluate d_p; d_p depends only on d, and $1 \leqslant d_p \leqslant d$ for all systems considered (with the equality holding only for $d = 1$), which is consistent with the intuitive notion that the incipient gel molecule is highly ramified. Finally, we made connection with the scaling and extended universality hypotheses. In particular, we found that the inequality $1 \leqslant d_p \leqslant d$ places bounds on the scaling powers and certain critical exponents, and that Suzuki's renormalised critical exponents all assume remarkably simple expressions when written in terms of d_p; in particular, the mean size exponent is simply d_p and the order parameter and 'decay of correlation' exponents are simply the co-dimension $d - d_p$.

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